2024 January Numerical Analysis Qualifer Solutions

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Problem 1

Let T be the unit triangle in $\mathbb{R}^2$ with vertices $v_1 = (0,0)$, $v_2 = (1,0)$, and $v_3 = (0,1)$. Denote the edges of T as $e_1 = v_1v_2$, $e_2 = v_2v_3$, and $e_3 = v_3v_1$. Let z_i be the midpoint of edge e_i, and let $\vec{t_i}$ be the counterclockwise pointing unit vector tangent to ∂T on e_i. Let TW_0 be the space of all vector-valued functions $\vec{p} : T \to \mathbb{R}^2$ of the form $\vec{p}(x,y) = (a - cy, b + cx)$ for some $a, b, c \in \mathbb{R}$. Then $\mathbb{P}_0^2 \subset TW_0 \subset \mathbb{P}_1^2$. Let $\sigma_i : TW_0 \to \mathbb{R}$ be defined by $\sigma_i(\vec{p}) = \vec{p}(z_i) \cdot \vec{t_i}$ for $i \in \{1,2,3\}$ and $\vec{p} \in TW_0$, and let $\Sigma = \{\sigma_1, \sigma_2, \sigma_3\}$.

Part a

Show that (T, TW_0, Σ) is a finite element triple.

Proof. Following Definition 5.2 in Ern and Guermond [1], it suffices to show that, given an arbitrary $\vec{p} \in TW_0$, if $\sigma_i(\vec{p}) = 0$ for each i, then $\vec{p} = \vec{0}$. Since $\vec{p} \in TW_0$, there are $a, b, c \in \mathbb{R}$ such that $\vec{p}(x,y) = (a - cy, b + cx)$ for each $(x,y) \in T$. Then since $z_1 = (1/2, 0)$ and $\vec{t_1} = (1,0)$, we have that

$$\sigma_1(\vec{p}) = \vec{p}(1/2, 0) \cdot (1,0) = (a, b + c/2) \cdot (1,0) = a = 0.$$

Similarly, since $z_3 = (0, 1/2)$ and $\vec{t_3} = (0,-1)$, we have that

$$\sigma_3(\vec{p}) = \vec{p}(0, 1/2) \cdot (0,-1) = (a - c/2, b) \cdot (0,-1) = -b = 0.$$

Finally, since $z_2 = (1/2, 1/2)$ and $\vec{t_2} = (-1,1)/\sqrt{2}$, we have that

$$\begin{aligned}\sigma_2(\vec{p}) = \vec{p}(1/2,1/2) \cdot (-1,1)/\sqrt{2} &= (a - c/2, b + c/2) \cdot (-1,1)/\sqrt{2} \\ &= (b + c - a)/\sqrt{2} = 0.\end{aligned}$$

Thus a, b, c satisfy the linear system

$$\begin{aligned} a &= 0, \\ b &= 0, \\ -a + b + c &= 0, \end{aligned}$$

which implies $a = b = c = 0$. This in turn implies $\vec{p} = \vec{0}$, which completes the proof. □

Part b

Find a basis $\{\vec{\varphi}_1, \vec{\varphi}_2, \vec{\varphi}_3\}$ of TW_0 that is dual to Σ. That is, $\sigma_i(\vec{\varphi}_j) = \delta_{ij}$ with $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$.

Proof. Since each $\vec{\varphi}_i \in TW_0$, there are $a_i, b_i, c_i \in \mathbb{R}$ such that $\vec{\varphi}_i(x, y) = (a_i - c_i y, b_i + c_i x)$ for all $(x, y) \in T$. Reusing some of the computations from the previous part then tells us that

$$\begin{aligned} \sigma_1(\vec{\varphi}_i) &= a_i = \delta_{i1}, \\ \sigma_3(\vec{\varphi}_i) &= -b_i = \delta_{i3}, \\ \sigma_2(\vec{\varphi}_i) &= (b_i + c_i - a_i)/\sqrt{2} = \delta_{i2}. \end{aligned}$$

Solving these explicitly for a_i, b_i, and c_i gives us

$$\begin{aligned} \vec{\varphi}_1(x, y) &= (1 - y, x), \\ \vec{\varphi}_2(x, y) &= \sqrt{2}(-y, x), \\ \vec{\varphi}_3(x, y) &= (y, x - 1) \end{aligned}$$

for all $(x, y) \in T$. □

Part c

Let $(\Pi\vec{u})(x, y) = \sum_{i=1}^{3} \sigma_i(\vec{u})\vec{\varphi}_i(x, y)$ for $(x, y) \in T$ and $\vec{u} \in H^2(T)^2$. Show that there exists $C > 0$ such that

$$\|\vec{u} - \Pi\vec{u}\|_{L^2(T)^2} \leq C(|\vec{u}|_{H^1(T)^2} + |\vec{u}|_{H^2(T)^2})$$

for all $\vec{u} \in H^2(T)^2$. You can use standard analysis results like trace, Sobolev, Poincaré inequalities and the Bramble-Hilbert Lemma without proof, but state precisely which results you are using.

Proof. We follow part of an argument from Theorem 11.13 in Ern and Guermond [1] adapted to this special case. We will make use of the Sobolev Embedding Theorem (Theorem 2.31 in Ern and Guermond [1]) as well as a Poincaré inequality found in Lemma 3.24 of Ern and Guermond [1]. We will explicitly restate these results as needed.

We first observe that for any $\vec{p} \in TW_0$, $\Pi\vec{p} = \vec{p}$. Indeed, this follows from the previous two parts, since any $\vec{p} \in TW_0$ can be expanded in the basis $\vec{\varphi}_i$ with its coefficients given by $\sigma_i(\vec{p})$. In other words, TW_0 is pointwise invariant under Π.

Next, we recall the following Sobolev inequality which states that $H^2(T)^2$ continuously embeds into $C^0(T)^2$, which is a consequence of Theorem 2.31 in Ern and Guermond [1]. Therefore, there is a constant $C_0 > 0$ such that

$$\max_{(x,y)\in T} |\vec{v}(x, y)| \leq C_0 \|\vec{v}\|_{H^2(T)^2} \tag{1}$$

for all $\vec{v} \in H^2(T)^2$. In particular, this implies that functions in $H^2(T)^2$ are continuous and bounded on T, and that

$$|\vec{v}(z_i)| \le C_0 \|\vec{v}\|_{H^2(T)^2} \tag{2}$$

for all $\vec{v} \in H^2(T)^2$.

Next, we show that there is a constant $C_1 > 0$ such that

$$\|\vec{v} - \Pi\vec{v}\|_{L^2(T)^2} \le C_1 \|\vec{v}\|_{H^2(T)^2} \tag{3}$$

for all $\vec{v} \in H^2(T)^2$. For this, we have that

$$\begin{aligned}
\|\vec{v} - \Pi\vec{v}\|_{L^2(T)^2} &\le \|\vec{v}\|_{L^2(T)^2} + \|\Pi\vec{v}\|_{L^2(T)^2}, \\
&\le \|\vec{v}\|_{H^2(T)^2} + \sum_{i=1}^{3} |\sigma_i(\vec{v})| \|\vec{\varphi}_i\|_{L^2(T)^2} \\
&\le \|\vec{v}\|_{H^2(T)^2} + (\max_i \|\vec{\varphi}_i\|_{L^2(T)^2}) \sum_i |\vec{v}(z_i) \cdot \vec{t}_i| \\
&\le \|\vec{v}\|_{H^2(T)^2} + (\max_i \|\vec{\varphi}_i\|_{L^2(T)^2}) \underbrace{(\max_i |\vec{t}_i|)}_{=1} \sum_i |\vec{v}(z_i)| \\
&\le (1 + 3C_0 \max_i \|\vec{\varphi}_i\|_{L^2(T)^2}) \|\vec{v}\|_{H^2(T)^2},
\end{aligned}$$

where in the first and second lines we used the triangle inequality, in the 4th line we used Cauchy-Schwarz, and in the last line we used the Sobolev inequality (2) from above. This shows the claim with $C_1 = 1 + 3C_0 \max_i \|\vec{\varphi}_i\|_{L^2(T)^2}$.

Now let $\vec{u} \in H^2(T)^2$ be arbitrary and let $\vec{p} \in \mathbb{P}_0^2$. Set $\vec{v} = \vec{u} - \vec{p}$. Since $\vec{p} \in \mathbb{P}_0^2 \subset TW_0$ and TW_0 is pointwise invariant under Π, we have that $\Pi\vec{p} = \vec{p}$. Then $\Pi\vec{v} = \Pi\vec{u} - \Pi\vec{p} = \Pi\vec{u} - \vec{p}$, so that

$$\vec{v} - \Pi\vec{v} = \vec{u} - \vec{p} - (\Pi\vec{u} - \vec{p}) = \vec{u} - \Pi\vec{u}.$$

Therefore, we apply the previous claim (3) to $\vec{v}$ and get

$$\|\vec{u} - \Pi\vec{u}\|_{L^2(T)^2} = \|\vec{v} - \Pi\vec{v}\|_{L^2(T)^2} \le C_1 \|\vec{v}\|_{H^2(T)^2} = C_1 \|\vec{u} - \vec{p}\|_{H^2(T)^2}. \tag{4}$$

Since $\vec{p} \in \mathbb{P}_0^2$, we have that

$$\|\vec{u} - \vec{p}\|_{H^2(T)^2}^2 = \|\vec{u} - \vec{p}\|_{L^2(T)^2}^2 + |\vec{u}|_{H^1(T)^2}^2 + |\vec{u}|_{H^2(T)^2}^2,$$

so that

$$\|\vec{u} - \vec{p}\|_{H^2(T)^2} \le \|\vec{u} - \vec{p}\|_{L^2(T)^2} + |\vec{u}|_{H^1(T)^2} + |\vec{u}|_{H^2(T)^2}. \tag{5}$$

Combining this with the previous inequality (4) gives us

$$\|\vec{u} - \Pi\vec{u}\|_{L^2(T)^2} \le C_1 \|\vec{u} - \vec{p}\|_{L^2(T)^2} + C_1 (|\vec{u}|_{H^1(T)^2} + |\vec{u}|_{H^2(T)^2}) \tag{6}$$

for all $\vec{u} \in H^2(T)^2$ and any $\vec{p} \in \mathbb{P}_0^2$.

Now we recall the Poincaré inequality, which states that there is a constant $C_2 > 0$ such that

$$\|\vec{u} - \underline{\vec{u}}\|_{L^2(T)^2} \leq C_2 |\vec{u}|_{H^1(T)^2}$$

for all $\vec{u} \in H^2(T)^2$, where

$$\underline{\vec{u}} = \frac{1}{|T|} \int_T \vec{u}(x, y)\, dx\, dy$$

is the average of $\vec{u}$ over T, which belongs to $\mathbb{P}_0^2$. Therefore, by taking $\vec{p} = \underline{\vec{u}}$ in the previous inequality (6) and using the Poincaré inequality, we have that

$$\begin{aligned} \|\vec{u} - \Pi\vec{u}\| &\leq C_1 C_2 |\vec{u}|_{H^1(T)^2} + C_1(|\vec{u}|_{H^1(T)^2} + |\vec{u}|_{H^2(T)^2}) \\ &\leq C(|\vec{u}|_{H^1(T)^2} + |\vec{u}|_{H^2(T)^2}) \end{aligned}$$

where $C = C_1(C_2 + 1)$ is independent of $\vec{u} \in H^2(T)^2$. Since $\vec{u}$ is arbitrary, we are done. □

References

[1] Alexandre Ern and Jean-Luc Guermond. *Finite Elements I: Approximation and Interpolation*. Springer, Switzerland, 2021.