Structure-Preserving Finite Element Schemes for the Euler-Poisson Equations

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Goal

Briefly describe a structure-preserving finite element scheme for the Euler-Poisson equations.

The Euler-Poisson Equations

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\begin{aligned}
\partial_t \rho + \nabla \cdot \mathbf{m} &= 0 \\
\partial_t \mathbf{m} + \nabla \cdot (\rho^{-1} \mathbf{m}\mathbf{m}^{\mathsf{T}} + Ip) &= -\rho \nabla \varphi \\
\partial_t E + \nabla \cdot (\rho^{-1} \mathbf{m}(E + p)) &= -\mathbf{m} \cdot \nabla \varphi \\
-\Delta \varphi &= \alpha \rho
\end{aligned}
$$

$\rho(\mathbf{x}, t) > 0$ mass density
$\mathbf{m}(\mathbf{x}, t) \in \mathbb{R}^d$ momentum
$E(\mathbf{x}, t) > 0$ total energy
$p = (\gamma - 1)(E - |\mathbf{m}|^2/(2\rho))$ pressure given by an equation of state
$\varphi(\mathbf{x}, t) \in \mathbb{R}$ scalar potential
$\alpha \in \mathbb{R}$ coupling constant

Structure

- Positivity of density: $\rho > 0$
- Positivity of internal energy: $e = E - |\mathbf{m}|^2/(2\rho) > 0$
- Local minimum principle on the specific entropy: $s \geq \min_{x\in\Omega} s_0(x)$, $s = e\rho^{-\gamma}$
- Formal energy balance:

$$\frac{d}{dt}\int_\Omega E + \frac{1}{2\alpha}|\nabla\varphi|^2\,dx = \text{boundary terms}$$

Operator Splitting

Strang / Yanenko operator splitting:

1. Given a discrete state $(\mathbf{u}^n = (\rho^n, \mathbf{m}^n, E^n), \varphi^n)$, compute a partial update $\mathbf{u}^{n+1,1}$ by discretizing and solving the Euler subsystem

$$\partial_t \mathbf{u} + \nabla \cdot F(\mathbf{u}) = 0$$

$$F(\mathbf{u}) = \begin{bmatrix} \mathbf{m}^\mathsf{T} \\ \rho^{-1}\mathbf{m}\mathbf{m}^\mathsf{T} + Ip \\ \rho^{-1}\mathbf{m}^\mathsf{T}(E+p) \end{bmatrix}$$

2. Use the partial update $(\mathbf{u}^{n+1,1}, \varphi^n)$ to compute the full update $(\mathbf{u}^{n+1}, \varphi^{n+1})$ by discretizing and solving the source-dominated subsystem

$$\partial_t \mathbf{u} = \begin{bmatrix} 0 \\ -\rho\nabla\varphi \\ -\mathbf{m}\cdot\nabla\varphi \end{bmatrix}$$

$$\partial_t \Delta\varphi = \alpha\nabla\cdot\mathbf{m}$$

Finite Elements

- Possibly non-affine quadrilateral(hexahedral) mesh
- Continuous bi(tri)-linear nodal finite element space $\mathbb{H}_h$ with basis $\{\chi_i\}$
- Discontinuous bi(tri)-linear nodal finite element space $\mathbb{V}_h$ with basis $\{\psi_i\}$
- Discretize $\rho_h^n = \sum_i \varrho_i^n \psi_i$, $\mathbf{m}_h^n = \sum_i \mathbf{M}_i^n \psi_i$, $E_h^n = \sum_i \mathcal{E}_i^n \psi_i$, $\mathbf{v}_h^n = \sum_i \mathbf{V}_i^n \psi_i$, $\varrho_i^n \mathbf{V}_i^n = \mathbf{M}_i^n$, $\varphi_h^n = \sum_i \Phi_i^n \chi_i$
- Lumped inner product

$$\langle f, g \rangle = \sum_K \sum_i f(\mathbf{x}_{K,i}) g(\mathbf{x}_{K,i}) \int_K \psi_{K,i} \, dx$$

Discretizing the Euler Subsystem

Graph viscosity and convex limiting (Guermond, Nazarov, Popov, Tomas 2018):

$$m_i \frac{\mathbf{u}_i^{n+1,1,L} - \mathbf{u}_i^n}{\tau_n} + \sum_j F(\mathbf{u}_j^n) \cdot \mathbf{c}_{ij} - d_{ij}^{n,L}(\mathbf{u}_j^n - \mathbf{u}_i^n) = 0$$

$$m_i = \int_\Omega \psi_i \, dx, \quad \mathbf{c}_{ij} = \int_\Omega \psi_i \nabla \psi_j \, dx$$

$$\sum_j m_{ij} \frac{\mathbf{u}_j^{n+1,1,H} - \mathbf{u}_j^n}{\tau_n} + F(\mathbf{u}_j^n) \cdot \mathbf{c}_{ij} - d_{ij}^{n,H}(\mathbf{u}_j^n - \mathbf{u}_i^n) = 0$$

$$\mathbf{u}_i^{n+1,1} = \mathbf{u}_i^{n+1,1,L} + \sum_j \ell_{ij}^n \mathbf{P}_{ij}^n$$

Discretizing the Source-Dominated System

Main challenge: discretize

$$\rho \partial_t \mathbf{v} = -\rho \nabla \varphi$$
$$\partial_t \Delta \varphi = \alpha \nabla \cdot (\rho \mathbf{v})$$

$$a^{\pm}_{\tau_n}(\varphi, \omega) = \int_\Omega \nabla \varphi \cdot \nabla \omega \, dx \pm \frac{\tau_n^2 \alpha}{4} \langle \rho_h^n \nabla \varphi, \nabla \omega \rangle$$

Given $(\mathbf{v}_h^n, \varphi_h^n) \in \mathbb{V}_h^d \times \mathbb{H}_h$, find $(\mathbf{v}_h^{n+1}, \varphi_h^{n+1}) \in \mathbb{V}_h^d \times \mathbb{H}_h$ for which

$$a^{+}_{\tau_n}(\varphi_h^{n+1}, \omega_h) = a^{-}_{\tau_n}(\varphi_h^n, \omega_h) + \tau_n \alpha \langle \rho_h^n \mathbf{v}_h^n, \nabla \omega_h \rangle$$
$$\langle \rho_h^n \mathbf{v}_h^{n+1}, \mathbf{z}_h \rangle = \langle \rho_h^n \mathbf{v}_h^n, \mathbf{z}_h \rangle - \frac{\tau_n}{2} \langle \rho_h^n (\nabla \varphi_h^{n+1} - \nabla \varphi_h^n), \mathbf{z}_h \rangle$$

for all $(\mathbf{z}_h, \omega_h) \in \mathbb{V}_h^d \times \mathbb{H}_h$

Main Result

Theorem (Maier, Tomas 2022)

Consider the Euler-Poisson equations on a bounded Lipschitz domain Ω with $\varphi = 0$ and $\mathbf{m} \cdot n = 0$ on $\partial\Omega$ and with prescribed initial conditions. Then the operator-splitting discretization scheme just described is invariant-domain preserving and satisfies a discrete version of the formal energy balance:

$$\sum_i m_i E_i^{n+1} + \frac{1}{2\alpha} \|\nabla \varphi_h^{n+1}\|_{L^2(\Omega)}^2 = \sum_i m_i E_i^n + \frac{1}{2\alpha} \|\nabla \varphi_h^n\|_{L^2(\Omega)}^2$$

Summary and Outlook

- We briefly described a structure-preserving finite element scheme for the Euler-Poisson equations
- Next step: couple the Euler equations with other PDEs (Maxwell's equations) and try to develop more structure-preserving schemes

References

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- J.-L.GUERMOND, M. NAZAROV, B. POPOV, I. TOMAS, Second-order invariant domain preserving approximation of the Euler equations using convex limiting, SIAM J. Sci. Comput., 40 5 (2018) A3211–A3239