Towards an involution-preserving solver for the time-dependent Maxwell equations

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Maxwell's equations and the related eigenvalue problem

Setting

- Open bounded Lipschitz polyhedron D in $\mathbb{R}^3$ with outward normal $\boldsymbol{n}_D$
- Magnetic field $\boldsymbol{B}(\boldsymbol{x}, t)$, electric field $\boldsymbol{E}(\boldsymbol{x}, t)$ on D
- No charges or currents; uniform material properties

Maxwell's equations with their involutions

$$\begin{aligned} \partial_t \boldsymbol{B} &= -\nabla \times \boldsymbol{E} \qquad & \nabla \cdot \boldsymbol{B} &= 0 \\ \partial_t \boldsymbol{E} &= \nabla \times \boldsymbol{B} \qquad & \nabla \cdot \boldsymbol{E} &= 0 \end{aligned}$$

Maxwell eigenvalue system in first-order form vs second-order form

$$\begin{aligned} -\nabla \times \boldsymbol{E} &= \lambda \boldsymbol{B} \\ \nabla \times \boldsymbol{B} &= \lambda \boldsymbol{E} \end{aligned} \quad \text{vs} \quad \nabla \times \nabla \times \boldsymbol{B} = -\lambda^2 \boldsymbol{B}$$

Goal Spectrally correct dG approximation of the first-order system

Theory

Precise formulation of the eigenvalue problem

Function spaces

$$\boldsymbol{L}^2(D) := \left\{\text{square integrable functions from } D \text{ to } \mathbb{C}^3\right\}$$
$$\boldsymbol{H}(\mathbf{curl}, D) := \left\{\boldsymbol{E} \in \boldsymbol{L}^2(D) \text{ with } \nabla \times \boldsymbol{E} \in \boldsymbol{L}^2(D)\right\}$$
$$\boldsymbol{H}_0(\mathbf{curl}, D) := \left\{\boldsymbol{B} \in \boldsymbol{H}(\mathbf{curl}, D) \text{ with } \boldsymbol{B} \times \boldsymbol{n}_D = \mathbf{0}\right\}$$
$$\boldsymbol{H}^c(D) := \boldsymbol{H}_0(\mathbf{curl}, D) \times \boldsymbol{H}(\mathbf{curl}, D)$$

Curl operators that are adjoint to eachother

$$\nabla\times : \boldsymbol{H}(\mathbf{curl}, D) \to \boldsymbol{L}^2(D) \qquad \nabla_0\times : \boldsymbol{H}_0(\mathbf{curl}, D) \to \boldsymbol{L}^2(D)$$

Eigenvalue problem Find all $\lambda \in \mathbb{C} \setminus \{0\}$, $(\boldsymbol{B}, \boldsymbol{E}) \in \boldsymbol{H}^c(D) \setminus \{(\mathbf{0}, \mathbf{0})\}$ such that

$$-\nabla \times \boldsymbol{E} = \lambda \boldsymbol{B} \qquad \nabla_0 \times \boldsymbol{B} = \lambda \boldsymbol{E}$$

Question Is the problem well-posed?

Formulating the involutions as orthogonality conditions

Observation

$$\begin{aligned} -\nabla \times \boldsymbol{E} &= \lambda \boldsymbol{B} \\ \nabla_0 \times \boldsymbol{B} &= \lambda \boldsymbol{E} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \boldsymbol{B} &\in \operatorname{im}(\nabla \times) \\ \boldsymbol{E} &\in \operatorname{im}(\nabla_0 \times) \end{aligned} \quad \Rightarrow \quad \begin{aligned} \nabla \cdot \boldsymbol{B} &= 0 \\ \nabla \cdot \boldsymbol{E} &= 0 \end{aligned}$$

Lemma (Ern, Guermond)

$$\boldsymbol{B} \in \operatorname{im}(\nabla \times) \Leftrightarrow \boldsymbol{B} \perp_{\boldsymbol{L}^2(D)} \underbrace{\ker(\nabla_0 \times)}_{\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0},D)} \qquad \boldsymbol{E} \in \operatorname{im}(\nabla_0 \times) \Leftrightarrow \boldsymbol{E} \perp_{\boldsymbol{L}^2(D)} \underbrace{\ker(\nabla \times)}_{\boldsymbol{H}(\mathbf{curl}=\mathbf{0},D)}$$

Definition (Involutions)

$$\boldsymbol{B} \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}, D)^{\perp_{\boldsymbol{L}^2(D)}} \qquad \boldsymbol{E} \in \boldsymbol{H}(\mathbf{curl} = \mathbf{0}, D)^{\perp_{\boldsymbol{L}^2(D)}}$$

Formulating a related boundary value problem

Idea

Formulate a well-posed boundary value problem with compact solution operator whose spectrum coincides with solutions to the eigenvalue problem

Orthogonal projections onto the kernels

$$\boldsymbol{\Pi}^c : \boldsymbol{L}^2(D) \to \boldsymbol{H}(\mathbf{curl} = \mathbf{0}, D) \qquad \boldsymbol{\Pi}^c_0 : \boldsymbol{L}^2(D) \to \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}, D)$$

Observation

$$\begin{aligned} -\nabla \times \boldsymbol{E} &= \lambda \boldsymbol{B} \\ \nabla_0 \times \boldsymbol{B} &= \lambda \boldsymbol{E} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \boldsymbol{\Pi}^c_0 \boldsymbol{B} &= \mathbf{0} \\ \boldsymbol{\Pi}^c \boldsymbol{E} &= \mathbf{0} \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \boldsymbol{B} &= (\boldsymbol{I} - \boldsymbol{\Pi}^c_0)\boldsymbol{B} \\ \boldsymbol{E} &= (\boldsymbol{I} - \boldsymbol{\Pi}^c)\boldsymbol{E} \end{aligned}$$

Boundary value problem

Given $\boldsymbol{f}, \boldsymbol{g} \in \boldsymbol{L}^2(D)$, find $\boldsymbol{B} \in \boldsymbol{H}_0(\mathbf{curl}, D)$, $\boldsymbol{E} \in \boldsymbol{H}(\mathbf{curl}, D)$ such that

$$\begin{aligned} -\nabla \times \boldsymbol{E} &= (\boldsymbol{I} - \boldsymbol{\Pi}^c_0)\boldsymbol{f} \qquad & \boldsymbol{\Pi}^c \boldsymbol{E} &= \mathbf{0} \\ \nabla_0 \times \boldsymbol{B} &= (\boldsymbol{I} - \boldsymbol{\Pi}^c)\boldsymbol{g} \qquad & \boldsymbol{\Pi}^c_0 \boldsymbol{B} &= \mathbf{0} \end{aligned}$$

Well-posedness of the eigenvalue problem

Lemma (Ern, Guermond)

Let $\boldsymbol{L}^c(D) = \boldsymbol{L}^2(D) \times \boldsymbol{L}^2(D)$ *and let* $\boldsymbol{H}^c(D) = \boldsymbol{H}_0(\mathbf{curl}, D) \times \boldsymbol{H}(\mathbf{curl}, D)$*. The boundary value problem is well-posed and the solution operator*

$$(\boldsymbol{f}, \boldsymbol{g}) \in \boldsymbol{L}^c(D) \mapsto T(\boldsymbol{f}, \boldsymbol{g}) := (\boldsymbol{B}, \boldsymbol{E}) \in \boldsymbol{H}^c(D) \hookrightarrow \boldsymbol{L}^c(D)$$

is compact.

Lemma (Ern, Guermond)

If $(\mu \neq 0, (\boldsymbol{B}, \boldsymbol{E}))$ *is an eigenpair of* T*, then* $(\lambda := 1/\mu, (\boldsymbol{B}, \boldsymbol{E}))$ *solves the eigenvalue problem.*
Conversely, if $(\lambda \neq 0, (\boldsymbol{B}, \boldsymbol{E}))$ *solves the eigenvalue problem, then* $(\mu := 1/\lambda, (\boldsymbol{B}, \boldsymbol{E}))$ *is an eigenpair of* T*.*

Discrete setting

Goal Approximate the spectrum of T.

Triangulations Shape-regular family of affine simplicial meshes $(\mathcal{T}_h)_h$ that cover D

Mesh faces $\mathcal{F}_h = (\text{interior faces } \mathcal{F}_h^\circ) \cup (\text{boundary faces } \mathcal{F}_h^\partial)$

Interfaces
For each interface $F \in \mathcal{F}_h^\circ$

- The two cells $K_{l,F}$, $K_{r,F}$ that share F
- Unit normal $\boldsymbol{n}_F$ on F pointing from $K_{l,F}$ to $K_{r,F}$
- Jump and average across F

$$[\![\boldsymbol{v}_h]\!]_F := \boldsymbol{v}_h|_{K_{l,F}} - \boldsymbol{v}_h|_{K_{r,F}} \qquad \{\!\{\boldsymbol{v}_h\}\!\}_F := \frac{\boldsymbol{v}_h|_{K_{l,F}} + \boldsymbol{v}_h|_{K_{r,F}}}{2}$$

Polynomial spaces $\mathbb{P}_k := \left\{\text{polynomials in 3 variables of degree at most } k\right\}$

Discrete spaces

$$\boldsymbol{P}_k^b(\mathcal{T}_h) := \left\{\boldsymbol{v}_h \in \boldsymbol{L}^2(D) : \boldsymbol{v}_h|_K \in \mathbb{P}_k^3 \text{ for all } K \in \mathcal{T}_h\right\} \qquad \boldsymbol{L}_h^c := \boldsymbol{P}_k^b(\mathcal{T}_h) \times \boldsymbol{P}_k^b(\mathcal{T}_h)$$

Discrete curls and fluxes

Discrete curl

$$(\nabla_h \times \boldsymbol{e}_h, \boldsymbol{b}_h)_{\boldsymbol{L}^2(D)} := \sum_{K \in \mathcal{T}_h} (\nabla \times (\boldsymbol{e}_h|_K), \boldsymbol{b}_h|_K)_{\boldsymbol{L}^2(K)}$$

$$\left(= (\nabla \times \boldsymbol{e}_h, \boldsymbol{b}_h)_{\boldsymbol{L}^2(D)} \text{ if } \boldsymbol{e}_h \in \boldsymbol{H}(\mathbf{curl}, D) \right)$$

Discrete fluxes

$$f_h^\circ(\boldsymbol{e}_h, \boldsymbol{b}_h) := \sum_{F \in \mathcal{F}_h^\circ} ([\![\boldsymbol{e}_h]\!]_F \times \boldsymbol{n}_F, \{\!\{\boldsymbol{b}_h\}\!\}_F)_{\boldsymbol{L}^2(F)} \qquad \left(= 0 \text{ if } \boldsymbol{e}_h \in \boldsymbol{H}(\mathbf{curl}, D) \right)$$

$$f_h(\boldsymbol{b}_h, \boldsymbol{e}_h) := f_h^\circ(\boldsymbol{b}_h, \boldsymbol{e}_h) + \sum_{F \in \mathcal{F}_h^\partial} (\boldsymbol{b}_h \times \boldsymbol{n}_F, \boldsymbol{e}_h)_{\boldsymbol{L}^2(F)} \qquad \left(= 0 \text{ if } \boldsymbol{b}_h \in \boldsymbol{H}_0(\mathbf{curl}, D) \right)$$

Consistency Terms

Consistency terms

$$C_h(\boldsymbol{e}_h, \boldsymbol{b}_h) := (\nabla_h \times \boldsymbol{e}_h, \boldsymbol{b}_h)_{\boldsymbol{L}^2(D)} + f_h^\circ(\boldsymbol{e}_h, \boldsymbol{b}_h)$$
$$C_{h0}(\boldsymbol{b}_h, \boldsymbol{e}_h) := (\nabla_h \times \boldsymbol{b}_h, \boldsymbol{e}_h)_{\boldsymbol{L}^2(D)} + f_h(\boldsymbol{b}_h, \boldsymbol{e}_h)$$

Lemma (Consistency and adjoints)

When $\boldsymbol{e}_h \in \boldsymbol{H}(\mathbf{curl}, D)$ *and* $\boldsymbol{b}_h \in \boldsymbol{H}_0(\mathbf{curl}, D)$,

$$C_h(\boldsymbol{e}_h, \boldsymbol{b}_h) = (\nabla \times \boldsymbol{e}_h, \boldsymbol{b}_h)_{\boldsymbol{L}^2(D)}$$
$$C_{h0}(\boldsymbol{b}_h, \boldsymbol{e}_h) = (\nabla_0 \times \boldsymbol{b}_h, \boldsymbol{e}_h)_{\boldsymbol{L}^2(D)}$$

For general $\boldsymbol{b}_h$, $\boldsymbol{e}_h$,

$$C_h(\boldsymbol{e}_h, \boldsymbol{b}_h)^* := \overline{C_h(\boldsymbol{b}_h, \boldsymbol{e}_h)} = C_{h0}(\boldsymbol{b}_h, \boldsymbol{e}_h)$$

Penalty terms

Penalize tangential jumps

$$s_h^{\circ}(\boldsymbol{E}_h, \boldsymbol{e}_h) := \sum_{F \in \mathcal{F}_h^{\circ}} ([\![\boldsymbol{E}_h]\!]_F \times \boldsymbol{n}_F, [\![\boldsymbol{e}_h]\!]_F \times \boldsymbol{n}_F)_{\boldsymbol{L}^2(F)}$$

Weakly enforce the boundary condition

$$s_h(\boldsymbol{B}_h, \boldsymbol{b}_h) := s_h^{\circ}(\boldsymbol{B}_h, \boldsymbol{b}_h) + \sum_{F \in \mathcal{F}_h^{\partial}} (\boldsymbol{B}_h \times \boldsymbol{n}_F, \boldsymbol{b}_h \times \boldsymbol{n}_F)_{\boldsymbol{L}^2(F)}$$

Lemma

If $\boldsymbol{E}_h \in \boldsymbol{H}(\mathbf{curl}, D)$, *then*

$$s_h^{\circ}(\boldsymbol{E}_h, \boldsymbol{e}_h) = 0.$$

Similarly, if $\boldsymbol{B}_h \in \boldsymbol{H}_0(\mathbf{curl}, D)$, *then*

$$s_h(\boldsymbol{B}_h, \boldsymbol{b}_h) = 0.$$

Discrete involutions

Involutions

$$\boldsymbol{B} \perp_{\boldsymbol{L}^2(D)} \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}, D) \qquad \boldsymbol{E} \perp_{\boldsymbol{L}^2(D)} \boldsymbol{H}(\mathbf{curl} = \mathbf{0}, D)$$

$$\boldsymbol{\Pi}_0^c : \boldsymbol{L}^2(D) \to \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}, D) \qquad \boldsymbol{\Pi}^c : \boldsymbol{L}^2(D) \to \boldsymbol{H}(\mathbf{curl} = \mathbf{0}, D)$$

$$\boldsymbol{\Pi}_0^c \boldsymbol{B} = \mathbf{0} \qquad \boldsymbol{\Pi}^c \boldsymbol{E} = \mathbf{0}$$

Discrete involutions

$$\boldsymbol{B}_h \perp_{\boldsymbol{L}^2(D)} \overbrace{\boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}, D) \cap \boldsymbol{P}_k^b(\mathcal{T}_h)}^{\boldsymbol{P}_{k0}^c(\mathbf{curl}=\mathbf{0},\mathcal{T}_h)} \qquad \boldsymbol{E}_h \perp_{\boldsymbol{L}^2(D)} \overbrace{\boldsymbol{H}(\mathbf{curl} = \mathbf{0}, D) \cap \boldsymbol{P}_k^b(\mathcal{T}_h)}^{\boldsymbol{P}_{k}^c(\mathbf{curl}=\mathbf{0},\mathcal{T}_h)}$$

$$\boldsymbol{\Pi}_{h0}^c : \boldsymbol{L}^2(D) \to \boldsymbol{P}_{k0}^c(\mathbf{curl} = \mathbf{0}, \mathcal{T}_h) \qquad \boldsymbol{\Pi}_h^c : \boldsymbol{L}^2(D) \to \boldsymbol{P}_k^c(\mathbf{curl} = \mathbf{0}, \mathcal{T}_h)$$

$$\boldsymbol{\Pi}_{h0}^c \boldsymbol{B}_h = \mathbf{0} \qquad \boldsymbol{\Pi}_h^c \boldsymbol{E}_h = \mathbf{0}$$

Discontinuous Galerkin formulation

Sesquilinear form

$$a_h((\boldsymbol{B}_h, \boldsymbol{E}_h), (\boldsymbol{b}_h, \boldsymbol{e}_h)) = C_{h0}(\boldsymbol{B}_h, \boldsymbol{e}_h) - C_h(\boldsymbol{E}_h, \boldsymbol{b}_h) + s_h(\boldsymbol{B}_h, \boldsymbol{b}_h) + s_h^{\circ}(\boldsymbol{E}_h, \boldsymbol{e}_h) + (\boldsymbol{\Pi}_{h0}^c \boldsymbol{B}_h, \boldsymbol{b}_h)_{\boldsymbol{L}^2(D)} + (\boldsymbol{\Pi}_h^c \boldsymbol{E}_h, \boldsymbol{e}_h)_{\boldsymbol{L}^2(D)}$$

Discrete eigenvalue problem

Find $\lambda_h \in \mathbb{C} \setminus \{0\}$, $(\boldsymbol{B}_h, \boldsymbol{E}_h) \in \boldsymbol{L}_h^c \setminus \{(\mathbf{0}, \mathbf{0})\}$ such that

$$a_h((\boldsymbol{B}_h, \boldsymbol{E}_h), (\boldsymbol{b}_h, \boldsymbol{e}_h)) = \lambda_h \Big((\boldsymbol{B}_h, \boldsymbol{b}_h)_{\boldsymbol{L}^2(D)} + (\boldsymbol{E}_h, \boldsymbol{e}_h)_{\boldsymbol{L}^2(D)} \Big)$$

for all $(\boldsymbol{b}_h, \boldsymbol{e}_h) \in \boldsymbol{L}_h^c$.

Discrete boundary value problem

Given $\boldsymbol{f}, \boldsymbol{g} \in \boldsymbol{L}^2(D)$, find $\boldsymbol{B}_h, \boldsymbol{E}_h \in \boldsymbol{P}_k^b(\mathcal{T}_h)$ such that

$$a_h((\boldsymbol{B}_h, \boldsymbol{E}_h), (\boldsymbol{b}_h, \boldsymbol{e}_h)) = ((\boldsymbol{I} - \boldsymbol{\Pi}_{h0}^c)\boldsymbol{f}, \boldsymbol{b}_h)_{\boldsymbol{L}^2(D)} + ((\boldsymbol{I} - \boldsymbol{\Pi}_h^c)\boldsymbol{g}, \boldsymbol{e}_h)_{\boldsymbol{L}^2(D)}$$

for all $\boldsymbol{b}_h, \boldsymbol{e}_h \in \boldsymbol{P}_k^b(\mathcal{T}_h)$.

Well-posedness of the discrete eigenvalue problem

Lemma (Ern, Guermond)

Let $\boldsymbol{L}^c(D) = \boldsymbol{L}^2(D) \times \boldsymbol{L}^2(D)$ *and let* $\boldsymbol{L}_h^c = \boldsymbol{P}_k^b(\mathcal{T}_h) \times \boldsymbol{P}_k^b(\mathcal{T}_h)$. *The discrete boundary value problem is well-posed, and the solution operator*

$$T_h : \boldsymbol{L}^c(D) \to \boldsymbol{L}_h^c \hookrightarrow \boldsymbol{L}^c(D)$$

is compact.

Lemma (Ern, Guermond)

If $(\mu_h \neq 0, (\boldsymbol{B}_h, \boldsymbol{E}_h))$ *is an eigenpair of* T_h, *then* $(\lambda_h := 1/\mu_h, (\boldsymbol{B}_h, \boldsymbol{E}_h))$ *solves the discrete eigenvalue problem.*
Conversely, if $(\lambda_h \neq 0, (\boldsymbol{B}_h, \boldsymbol{E}_h))$ *solves the discrete eigenvalue problem, then* $(\mu_h := 1/\lambda_h, (\boldsymbol{B}_h, \boldsymbol{E}_h))$ *is an eigenpair of* T_h.

The dG approximation is spectrally correct

Theorem (Ern, Guermond)

There exists $\sigma \in (0, 1/2)$ *and a constant* $C > 0$ *such that*

$$\|T - T_h\|_{\boldsymbol{L}^2(D) \times \boldsymbol{L}^2(D)} \leq Ch^{\sigma}$$

for all h.

Corollary

The discrete eigenvalue problem is a spectrally correct approximation of the continuous eigenvalue problem.
That is,

1. *the discrete eigenvalues converge to the continuous eigenvalues with the correct multiplicities,*
2. *the discrete eigenspaces converge to the continuous eigenspaces, and*
3. *there are no spurious eigenvalues or eigenfunctions.*

Numerics

The discrete projections do not have to be implemented

Lemma (Ern, Guermond 2023)

Let

$$\hat{a}_h((\boldsymbol{B}_h, \boldsymbol{E}_h), (\boldsymbol{b}_h, \boldsymbol{e}_h)) = a_h((\boldsymbol{B}_h, \boldsymbol{E}_h), (\boldsymbol{b}_h, \boldsymbol{e}_h)) - (\boldsymbol{\Pi}^c_{h0}\boldsymbol{B}_h, \boldsymbol{b}_h)_{\boldsymbol{L}^2(D)} - (\boldsymbol{\Pi}^c_h\boldsymbol{E}_h, \boldsymbol{e}_h)_{\boldsymbol{L}^2(D)}$$

Then $(\lambda_h \neq 0, (\boldsymbol{B}_h, \boldsymbol{E}_h))$ *satisfies*

$$a_h((\boldsymbol{B}_h, \boldsymbol{E}_h), (\boldsymbol{b}_h, \boldsymbol{e}_h)) = \lambda_h\Big((\boldsymbol{B}_h, \boldsymbol{b}_h)_{\boldsymbol{L}^2(D)} + (\boldsymbol{E}_h, \boldsymbol{e}_h)_{\boldsymbol{L}^2(D)}\Big)$$

for all $\boldsymbol{b}_h, \boldsymbol{e}_h \in \boldsymbol{P}^b_k(\mathcal{T}_h)$ *iff it satisfies*

$$\hat{a}_h((\boldsymbol{B}_h, \boldsymbol{E}_h), (\boldsymbol{b}_h, \boldsymbol{e}_h)) = \lambda_h\Big((\boldsymbol{B}_h, \boldsymbol{b}_h)_{\boldsymbol{L}^2(D)} + (\boldsymbol{E}_h, \boldsymbol{e}_h)_{\boldsymbol{L}^2(D)}\Big)$$

for all $\boldsymbol{b}_h, \boldsymbol{e}_h \in \boldsymbol{P}^b_k(\mathcal{T}_h)$.

The 2d problem

The 2d setting

Bounded open Lipschitz polygon D in $\mathbb{R}^2$
Outward normal $\boldsymbol{n}_D$ and tangential vector $\boldsymbol{t}_D := (n_2, -n_1)$

$$\boldsymbol{B}(\boldsymbol{x}, t) \in \mathbb{C}^2 \qquad E(\boldsymbol{x}, t) \in \mathbb{C}$$

$$\nabla \times \boldsymbol{B} := \mathrm{curl}\, \boldsymbol{B} := \partial_1 B_2 - \partial_2 B_1 \qquad \nabla \times E := \mathrm{rot}\, E := (\partial_2 E, -\partial_1 E)$$

$$\boldsymbol{B} \times \boldsymbol{n}_D := -\boldsymbol{B} \cdot \boldsymbol{t}_D \qquad E \times \boldsymbol{n}_D := E\boldsymbol{t}_D$$

$$\boldsymbol{L}^2(D) := L^2(D) \times L^2(D) \qquad L^2(D) := \left\{ v : D \to \mathbb{C} : \int_D |u|^2 < \infty \right\}$$

$$\boldsymbol{P}_k^b(\mathcal{T}_h) := P_k^b(\mathcal{T}_h) \times P_k^b(\mathcal{T}_h) \qquad P_k^b(\mathcal{T}_h) := \left\{ v_h \in L^2(D) : v_h|_K \in \mathbb{P}_k \; \forall K \in \mathcal{T}_h \right\}$$

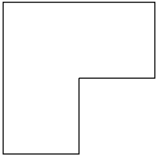
Sesquilinear form

$$\boldsymbol{L}_h^c := \boldsymbol{L}^2(D) \times L^2(D)$$

$$\hat{a}_h : \boldsymbol{L}_h^c \times \boldsymbol{L}_h^c \to \mathbb{C} \text{ defined exactly the same}$$

Convergence of the smallest eigenvalue with positive imaginary part on the L-shaped domain

Domain $D = (-1,1)^2 \setminus ((0,1) \times (-1,0))$



$\mathbb{P}_0$ piecewise constants

dofs	h	λ_1	Rel. Error	ROC
1152	1.8924E-01	8.3329E-02 + 1.2038i	6.9187E-02	
4608	9.4609E-02	4.2023E-02 + 1.2107i	3.4751E-02	0.99
18432	4.7318E-02	2.1105E-02 + 1.2132i	1.7420E-03	1.00
73728	2.3660E-02	1.0574E-02 + 1.2142i	8.7183E-04	1.00

Convergence of the smallest eigenvalue with positive imaginary part on the L-shaped domain

$\mathbb{P}_1$ polynomials

dofs	h	λ_1	Rel. Error	ROC
3456	1.8924E-01	1.8802E-04 + 1.2132i	1.2715E-03	
13824	9.4609E-02	3.5848E-05 + 1.2142i	4.9610E-04	1.36
55296	4.7318E-02	7.2335E-06 + 1.2145i	2.0432E-04	1.28
221184	2.3660E-02	1.4319E-06 + 1.2147i	8.1821E-05	1.32

$\mathbb{P}_2$ polynomials

dofs	h	λ_1	Rel. Error	ROC
6912	1.8924E-01	2.1445E-05 + 1.2141i	5.0330E-04	
27648	9.4609E-02	4.1952E-06 + 1.2145i	1.9755E-04	1.35
110592	4.7318E-02	8.8709E-07 + 1.2147i	8.1771E-05	1.27
442368	2.3660E-02	1.7644E-07 + 1.2147i	3.2725E-05	1.32

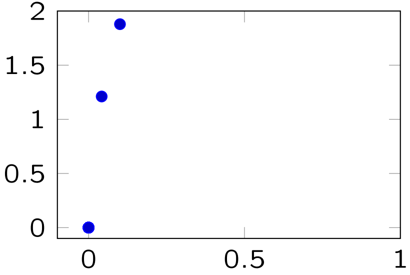
$\mathbb{Q}_k$ polynomials on quadrilateral meshes

Tensor-product polynomials

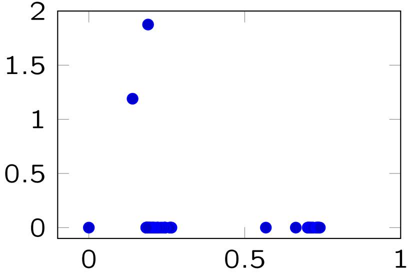
$$\mathbb{Q}_k := \{\text{polynomials in 3 variables of degree at most } k \text{ in each variable}\}$$

Presence of spurious modes for $\mathbb{Q}_k$ polynomials on quadrilateral meshes

$\mathbb{P}_0$, triangles, h = 9.4609E-02



$\mathbb{Q}_0$, quadrilaterals, h = 8.8388E-02



Conclusion

- The dG approximation of the Maxwell eigenvalue problem in first-order form is spectrally correct for $\mathbb{P}_k$ polynomials on simplicial meshes.
- Some very recent numerical experiments indicate that for tensor product $\mathbb{Q}_k$ polynomials on quadrilateral meshes, the method appears to have spurious modes.
- Further work is needed to determine the source of these spurious modes for the first-order system; there is literature about the second-order system.
- Future goals: solving the time-dependent Maxwell equations; multiphysics systems; the Euler-Maxwell equations.

References

- *The discontinuous Galerkin approximation of the grad-div and curl-curl operators in first-order form is involution-preserving and spectrally correct*, Alexandre Ern and Jean-Luc Guermond, 2023
- *Spectral correctness of the discontinuous Galerkin approximation of the first-order form of Maxwell's equations with discontinuous coefficients*, Alexandre Ern and Jean-Luc Guermond, 2023
- *The deal.II finite element library*, `https://www.dealii.org/`

Thank you!