A Comparison of Finite Element Spaces for the Discontinuous Galerkin Approximation of the Maxwell Eigenvalue Problem in First-Order Form

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Today

Question

How do we discretize Maxwell's equations in space using discontinuous finite elements?

$$\partial_t \mathbf{E} = c\nabla \times \mathbf{B}$$
$$\partial_t \mathbf{B} = -c\nabla \times \mathbf{E}$$

$$\nabla \cdot \mathbf{E} = 0$$
$$\nabla \cdot \mathbf{B} = 0$$

The Maxwell operator

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{0} & \nabla\times \\ -\nabla\times & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} \nabla \times \mathbf{B} \\ -\nabla \times \mathbf{E} \end{pmatrix}$$

Goal

Discretize this operator with discontinuous finite elements in a way that

1. preserves the **involutions**

$$\nabla \cdot \mathbf{E} = 0$$
$$\nabla \cdot \mathbf{B} = 0$$

2. and is **spectrally correct**

The Maxwell eigenvalue problem

Setting

$D \subset \mathbb{R}^3$ open, bounded, connected, and Lipschitz

$\mathbf{n}_D$ outward normal

$$\mathbf{H}(\mathrm{curl}, D) := \left\{\mathbf{e} \in L^2(D)^3 : \nabla \times \mathbf{e} \in L^2(D)^3\right\}$$

$$\mathbf{H}_0(\mathrm{curl}, D) := \left\{\mathbf{b} \in \mathbf{H}(\mathrm{curl}, D) : \mathbf{b} \times \mathbf{n}_D = \mathbf{0}\right\}$$

$$\nabla\times : \mathbf{H}(\mathrm{curl}, D) \to L^2(D)^3$$

$$\nabla_0\times : \mathbf{H}_0(\mathrm{curl}, D) \to L^2(D)^3$$

Problem

Find $\lambda \in \mathbb{C}$, $\mathbf{E} \in \mathbf{H}(\mathrm{curl}, D)$, $\mathbf{B} \in \mathbf{H}_0(\mathrm{curl}, D)$ such that

$$\nabla_0 \times \mathbf{B} = \lambda \mathbf{E}$$

$$-\nabla \times \mathbf{E} = \lambda \mathbf{B}$$

The spectrum

$\lambda = 0$ **(Bad)**
Unphysical, no involutions

$\lambda \neq 0$ **(Good)**
Involution-preserving:

$$
\begin{aligned}
\nabla_0 \times \mathbf{B} = \lambda \mathbf{E} \quad &\Rightarrow \quad \mathbf{E} \in \operatorname{im}(\nabla_0 \times) \quad \Rightarrow \quad \nabla \cdot \mathbf{E} = 0 \\
\nabla \times \mathbf{E} = \lambda \mathbf{B} \quad &\Rightarrow \quad \mathbf{B} \in \operatorname{im}(\nabla \times) \quad \Rightarrow \quad \nabla \cdot \mathbf{B} = 0
\end{aligned}
$$

Involution-preserving spaces

$$
\mathbf{E} \in \mathbf{X}^c := \mathbf{H}(\mathrm{curl}, D) \cap \operatorname{im}(\nabla_0 \times) = \mathbf{H}(\mathrm{curl}, D) \cap \ker(\nabla \times)^{\perp_{L^2}}
$$

$$
\mathbf{B} \in \mathbf{X}_0^c := \mathbf{H}_0(\mathrm{curl}, D) \cap \operatorname{im}(\nabla \times) = \mathbf{H}_0(\mathrm{curl}, D) \cap \ker(\nabla_0 \times)^{\perp_{L^2}}
$$

The spectrum

Theorem (A. Ern and J.-L. Guermond, 2023)

There is a compact operator S on $L^2(D)^3 \times L^2(D)^3$ such that $\lambda \neq 0$, $\mathbf{E} \in \mathbf{X}^c$, $\mathbf{B} \in \mathbf{X}_0^c$ solves the Maxwell eigenvalue problem iff $(1/\lambda, (\mathbf{E}, \mathbf{B}))$ is an eigenpair of S.

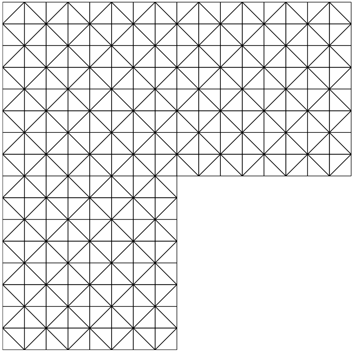
Remarks

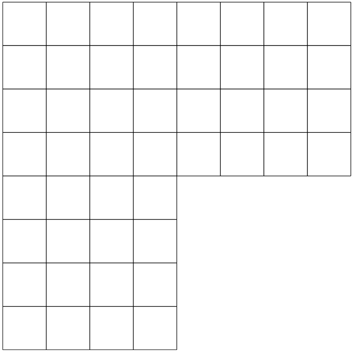
- The involutions are essential to showing this
- S is a solution operator to a related boundary value problem

Discontinuous Galerkin approximation

Meshes

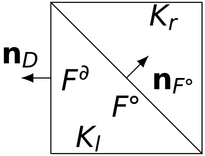
1. Reference cell $\widehat{K}$ (tetrahedron or cube)
2. Reference transformations $T_K : \widehat{K} \to K$ (affine or Cartesian)





Discontinuous Galerkin approximation

$$\nabla_0 \times \mathbf{B} = \lambda \mathbf{E}$$
$$-\nabla \times \mathbf{E} = \lambda \mathbf{B}$$



$$[\mathbf{B}] := \mathbf{B}|_{K_l} - \mathbf{B}|_{K_r}$$
$$\{\mathbf{B}\} := \frac{\mathbf{B}|_{K_l} + \mathbf{B}|_{K_r}}{2}$$

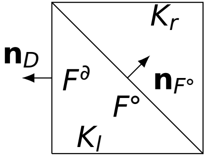
Test and integrate on a cell, sum over cells

$$\sum_K \int_K (\nabla_0 \times \mathbf{B}) \cdot \mathbf{e} - (\nabla \times \mathbf{E}) \cdot \mathbf{b} \, \mathrm{d}\mathbf{x}$$

$$= \lambda \sum_K \int_K \mathbf{E} \cdot \mathbf{e} + \mathbf{B} \cdot \mathbf{b} \, \mathrm{d}\mathbf{x}$$

Discontinuous Galerkin approximation

$$\nabla_0 \times \mathbf{B} = \lambda \mathbf{E}$$
$$-\nabla \times \mathbf{E} = \lambda \mathbf{B}$$



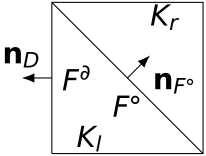
$$[\mathbf{B}] := \mathbf{B}|_{K_l} - \mathbf{B}|_{K_r}$$
$$\{\mathbf{B}\} := \frac{\mathbf{B}|_{K_l} + \mathbf{B}|_{K_r}}{2}$$

Add consistency terms

$$\sum_K \int_K (\nabla_0 \times \mathbf{B}) \cdot \mathbf{e} - (\nabla \times \mathbf{E}) \cdot \mathbf{b}\, \mathrm{d}\mathbf{x} + \sum_{F^\circ} \int_{F^\circ} [\mathbf{B}] \times \mathbf{n}_{F^\circ} \cdot \{\mathbf{e}\} - [\mathbf{E}] \times \mathbf{n}_{F^\circ} \cdot \{\mathbf{b}\}\, \mathrm{d}s$$
$$+ \sum_{F^\partial} \int_{F^\partial} \mathbf{B} \times \mathbf{n}_D \cdot \mathbf{e}\, \mathrm{d}s$$
$$= \lambda \sum_K \int_K \mathbf{E} \cdot \mathbf{e} + \mathbf{B} \cdot \mathbf{b}\, \mathrm{d}\mathbf{x}$$

Discontinuous Galerkin approximation

$$\nabla_0 \times \mathbf{B} = \lambda \mathbf{E}$$
$$-\nabla \times \mathbf{E} = \lambda \mathbf{B}$$



$$[\mathbf{B}] := \mathbf{B}|_{K_l} - \mathbf{B}|_{K_r}$$
$$\{\mathbf{B}\} := \frac{\mathbf{B}|_{K_l} + \mathbf{B}|_{K_r}}{2}$$

Add penalty terms $\quad a_h((\mathbf{E},\mathbf{B}),(\mathbf{e},\mathbf{b})) = \lambda m_h((\mathbf{E},\mathbf{B}),(\mathbf{e},\mathbf{b}))$

$$\sum_K \int_K (\nabla_0 \times \mathbf{B})\cdot \mathbf{e} - (\nabla \times \mathbf{E})\cdot \mathbf{b}\,\mathrm{d}\mathbf{x} + \sum_{F^\circ}\int_{F^\circ} [\mathbf{B}] \times \mathbf{n}_{F^\circ} \cdot \{\mathbf{e}\} - [\mathbf{E}] \times \mathbf{n}_{F^\circ} \cdot \{\mathbf{b}\}\,\mathrm{d}s$$
$$+ \sum_{F^\partial}\int_{F^\partial} \mathbf{B} \times \mathbf{n}_D \cdot \mathbf{e}\,\mathrm{d}s + \sum_{F^\circ}\int_{F^\circ} ([\mathbf{B}] \times \mathbf{n}_F)\cdot([\mathbf{b}] \times \mathbf{n}_F) + ([\mathbf{E}] \times \mathbf{n}_F)\cdot([\mathbf{e}] \times \mathbf{n}_F)\,\mathrm{d}s$$
$$+ \sum_{F^\partial}\int_{F^\partial} (\mathbf{B} \times \mathbf{n}_D)\cdot(\mathbf{b} \times \mathbf{n}_D)\,\mathrm{d}s = \lambda \sum_K \int_K \mathbf{E}\cdot\mathbf{e} + \mathbf{B}\cdot\mathbf{b}\,\mathrm{d}\mathbf{x}$$

Discrete eigenvalue problem

Find $\lambda_h \in \mathbb{C}$ and $\mathbf{E}_h, \mathbf{B}_h \in \mathbf{P}^b(\mathcal{T}_h, \widehat{\mathbf{P}})$ such that

$$a_h((\mathbf{E}_h, \mathbf{B}_h), (\mathbf{e}_h, \mathbf{b}_h)) = \lambda_h m_h((\mathbf{E}_h, \mathbf{B}_h), (\mathbf{e}_h, \mathbf{b}_h))$$

for all test functions $\mathbf{e}_h, \mathbf{b}_h \in \mathbf{P}^b(\mathcal{T}_h, \widehat{\mathbf{P}})$, where

$$\mathbf{P}^b(\mathcal{T}_h, \widehat{\mathbf{P}}) := \left\{\mathbf{e}_h \in L^2(D)^3 : \mathbf{e}_h \circ T_K \in \widehat{\mathbf{P}} \text{ for all } K \in \mathcal{T}_h\right\}$$

and $\widehat{\mathbf{P}}$ is a space of vector-valued polynomials on $\widehat{K}$

Polynomial spaces

Simplicial meshes

- $\widehat{\mathbf{P}} = \mathbb{P}_k^3$ vector-valued polynomials total degree at most $k \geq 0$

Cartesian hexahedral meshes

- $\widehat{\mathbf{P}} = \mathbb{Q}_k^3$ vector-valued polynomials total degree at most $k \geq 0$ in each variable
- $\widehat{\mathbf{P}} = \mathbb{N}_k^3 := \mathbb{Q}_{k,k+1,k+1} \times \mathbb{Q}_{k+1,k,k+1} \times \mathbb{Q}_{k+1,k+1,k}$ Cartesian Nédélec polynomials of the first kind
- $\widehat{\mathbf{P}} = \mathbb{Q}_{k,\mathrm{curl}}^3 := \mathbb{Q}_k^3 + \nabla \mathbb{Q}_{k+1}$

Spectral Correctness

$(\lambda, (\mathbf{E}, \mathbf{B}))$ exact eigenpairs, $(\lambda_h, (\mathbf{E}_h, \mathbf{B}_h))$ discrete eigenpairs

The approximation is **spectrally correct** if

1. eigenvalues $\lambda_h \to \lambda$ (with correct multiplicity)
2. eigenspaces $E(\lambda_h) \to E(\lambda)$ (subspace gap)
3. no spurious eigenvalues (numerical garbage)

D. Boffi
Finite element approximation of eigenvalue problems
Article, Acta Numerica, 2010

Spectral Correctness

Question

Which polynomial spaces give a spectrally correct approximation?

Theorem (A. Ern and J.-L. Guermond, 2023)

Affine simplicial meshes with $\mathbb{P}_k^3$ polynomials give a spectrally correct approximation.

Theorem

$\mathbb{Q}_k^3$ polynomials are spurious.

Theorem

Cartesian hexahedral meshes with $\mathbb{N}_k^3$ or $\mathbb{Q}_{k,\mathrm{curl}}^3$ polynomials also give a spectrally correct approximation.

Spectral Correctness

Remarks about the proof

1. Discrete versions of the involutions (being orthogonal to enough gradients) must be strong enough to establish spectral correctness
2. $\mathbb{Q}_k^3$ is spurious because it does not contain enough gradients

$$\nabla \mathbb{Q}_{k+1} \not\subset \mathbb{Q}_k^3$$

$$\nabla \mathbb{P}_{k+1} \subset \mathbb{P}_k^3$$

$$\nabla \mathbb{Q}_{k+1} \subset \mathbb{N}_k^3$$

$$\nabla \mathbb{Q}_{k+1} \subset \mathbb{Q}_k^3 + \nabla \mathbb{Q}_{k+1} := \mathbb{Q}_{k,\text{curl}}^3$$

Numerical experiments

- 2D test problems, formally take $\mathbf{E} = (0, 0, E_z)$ and $\mathbf{B} = (B_x, B_y, 0)$
- deal.II finite element library for assembly
- ARPACK to solve the matrix-vector generalized eigenvalue problem

$$\mathbf{A}_h \mathbf{x}_h = \lambda_h \mathbf{M}_h \mathbf{x}_h$$

- Goal: approximate the smallest nonzero eigenvalues

Numerical experiments

Test 1: unit square

$$\nabla_0 \cdot \mathbf{B}^\perp = \lambda E$$
$$-\nabla^\perp E = \lambda \mathbf{B}$$

Eigenvalues

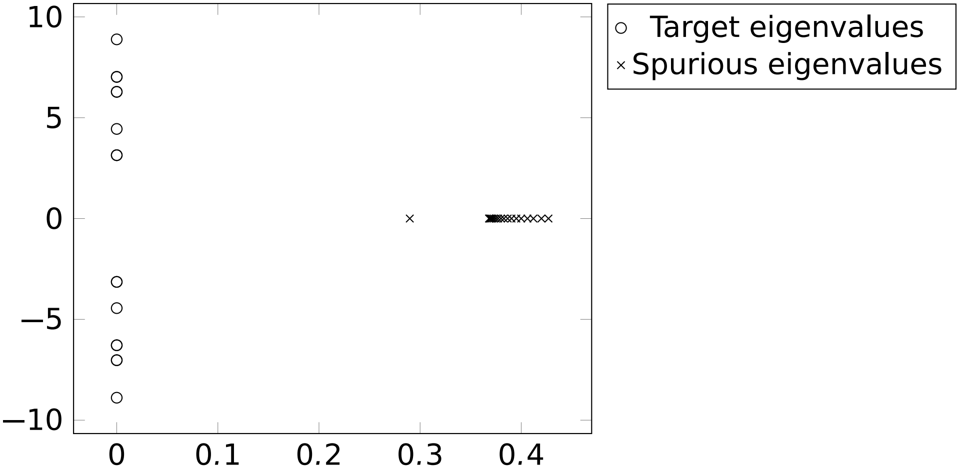
$$\lambda_{j,k} = \pm i\pi\sqrt{j^2 + k^2}$$

Eigenfunctions

$$\mathbf{B}_{j,k} = \left(-i\frac{k}{\sqrt{j^2+k^2}}\cos(j\pi x)\sin(k\pi y),\ i\frac{j}{\sqrt{j^2+k^2}}\sin(j\pi x)\cos(k\pi y)\right)$$
$$E_{j,k} = \cos(j\pi x)\cos(k\pi y)$$

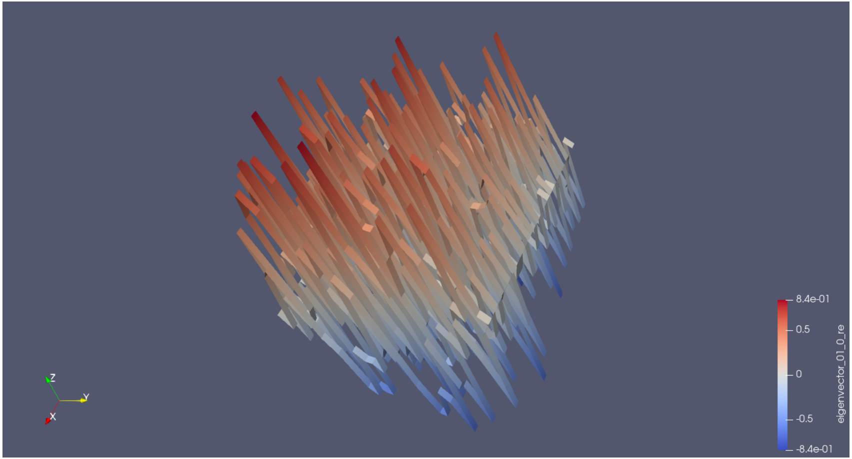
Numerical experiments

Spurious eigenvalues for $\mathbb{Q}_k^2$

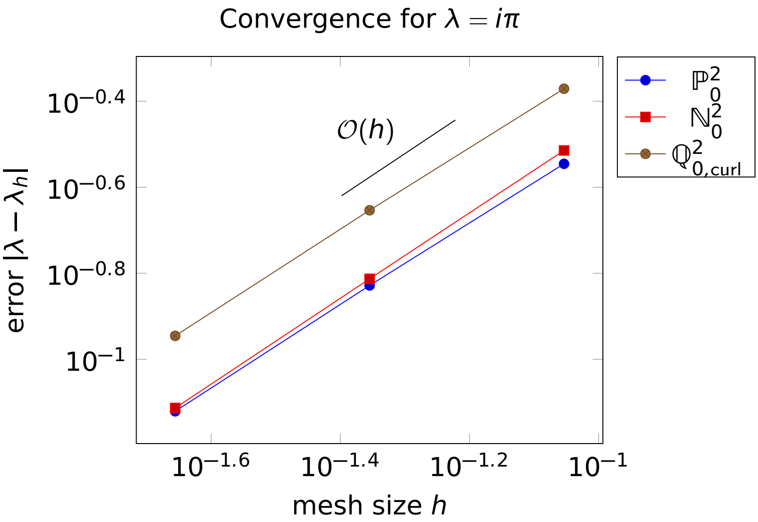


Numerical experiments

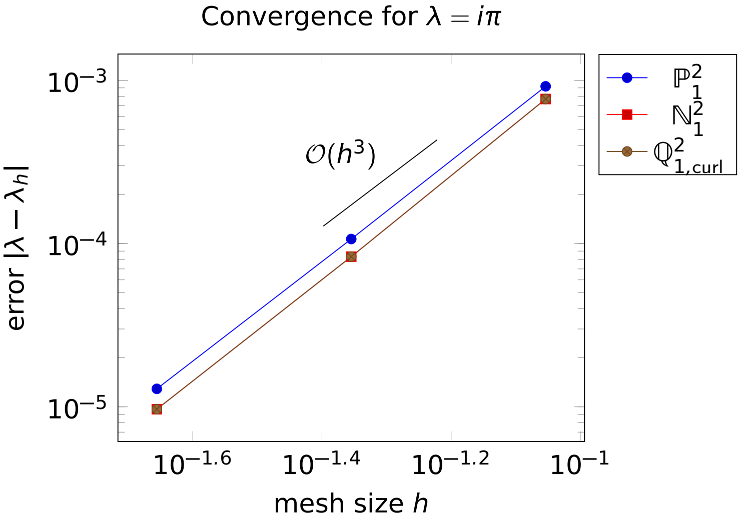
Spurious eigenfunction for $\mathbb{Q}_k^2$



Numerical experiments

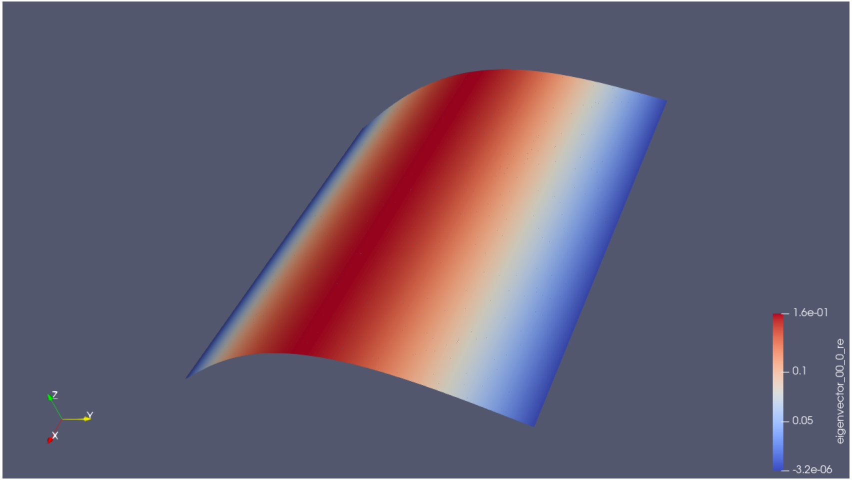


Numerical experiments



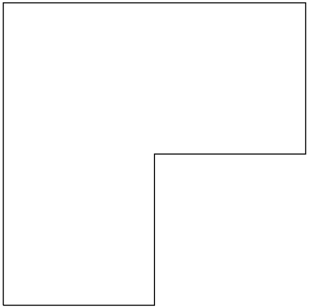
Numerical experiments

Spectrally correct eigenfunction for $\lambda = i\pi$



Numerical experiments

Test 2: L-shaped domain

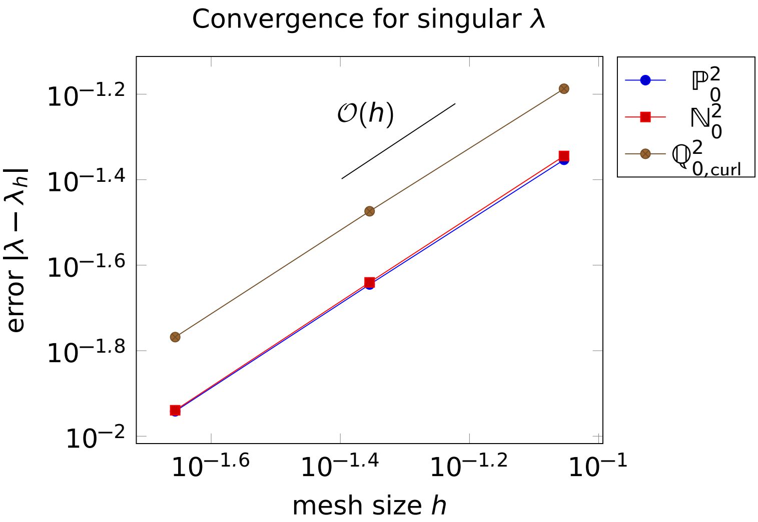


Eigenfunctions can become singular at the re-entrant corner!

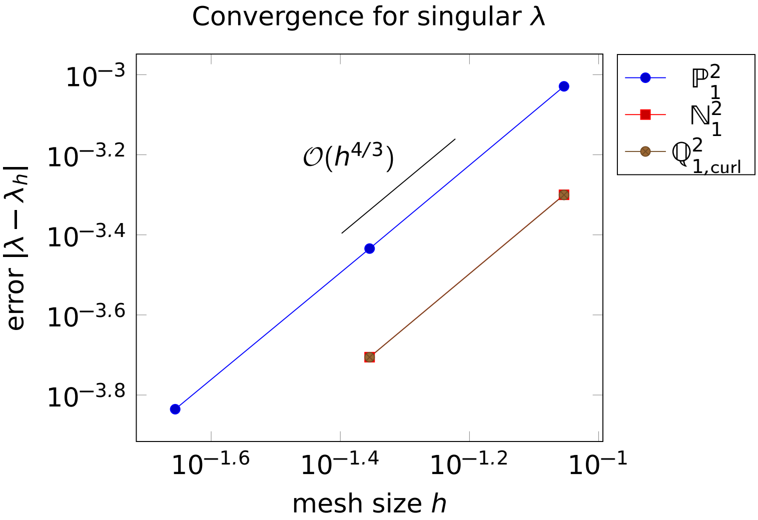
Imaginary parts of smallest eigenvalues (from M. Dauge)

singular! → **1.214751754**
1.879901957
3.141592654
3.141592654
3.374830277

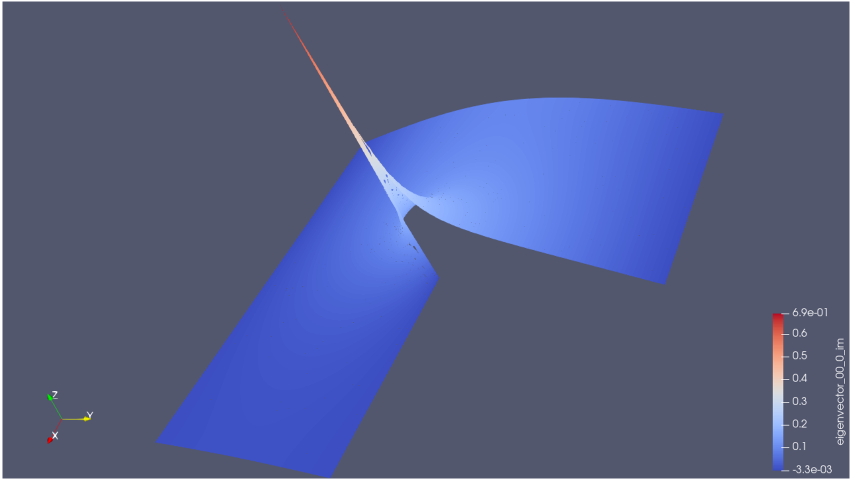
Numerical experiments



Numerical experiments



Numerical experiments
Singular eigenfunction



Conclusion

1. When discretizing the Maxwell operator with discontinuous finite elements
   - $\mathbb{P}_k^3$ polynomials on affine simplicial meshes are spectrally correct
   - $\mathbb{Q}_k^3$ polynomials are spurious
   - $\mathbb{N}_k^3$ and $\mathbb{Q}_{k,\text{curl}}^3$ polynomials on Cartesian hexahedral meshes are spectrally correct

2. The spectrally correct spaces can obtain optimal error rates for the related eigenvalue problem

3. TODOs
   - Non-affine meshes?
   - Proofs for error estimates?
   - 3d simulations?
   - Time-dependent Maxwell?

Further reading

- A. Ern and J.-L. Guermond
  Spectral correctness of the discontinuous Galerkin approximation of the first-order form of Maxwell's equations with discontinuous coefficients
  Preprint, https://hal.science/hal-04145808, 2024

- V. Perrier
  Discrete de-Rham complex involving a discontinuous finite element space for velocities: the case of periodic straight triangular and Cartesian meshes
  Preprint, https://arxiv.org/abs/2404.19545, 2024

- J. Hoffart
  A comparison of finite element spaces for the discontinuous Galerkin approximation of the Maxwell eigenvalue problem in first-order form
  Preprint, 2024